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A New Second-Order Shallow-Water Scheme on Two-Dimensional Structured Grids based on Hydrostatic Reconstruction

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Abstract

This poster presents a finite volume (FV) scheme on structured grids to simulate shallow flows over complex terrain. The new second-order scheme is based on a recent first-order hydrostatic reconstruction method, which has an improved handling of shallow flows over steps.
The scheme is well-balanced, positivity-preserving, and handles dry cells. In wet-dry transition zones and partially wet cells the new scheme differs from the classical second-order hydrostatic reconstruction scheme, in contrast to fully wet regions where the two schemes coincide. The scheme is particularly stable due to a strong-stability-preserving Runge-Kutta time integration, which efficiently suppresses oscillations. For shallow flows over discontinuous bottom steps the new scheme produces better results than the original scheme. Furthermore, we test the scheme on various benchmark tests to assess accuracy and robustness, such as the U-shaped flume and the parabolic bump. The method produces comparable results to other HR-based schemes.

The Shallow-Water Equations (SWEs)

The SWEs serve as a basis for the modeling of flood simulations. They are obtained by depth integrating the Euler equations using the shallow-water assumption. In two spatial dimensions they read

$$\begin{aligned}
\partial_t h &+ \partial_x (hu) &+ \partial_y (hv) & &= 0,\\
\partial_t (hu) &+ \partial_x (hu^2 + gh^2/2) &+ \partial_y (huv) &+ gh\partial_x b &= 0,\\
\partial_t (hv) &+ \partial_x (huv) &+ \partial_y (hv^2 + gh^2/2) &+ gh\partial_y b &= 0.
\end{aligned}$$

The state vector $U = (h, hu, hv)$ consists of the water surface depth h and the discharges hu and hv in x and y dimension, respectively.
In two dimensions, the SWEs allow for complicated steady state solutions, however we restrict ourselves to two important steady-state equilibria. Following Chen and Noelle (2017), there is the still-water equilibrium, i. e.,

$$u, v = 0, \quad \text{and} \quad \partial_x w, \partial_y w = 0,$$

where w denotes the water level $w = h + b$, and the lake at rest equilibrium, which includes dry shores, i. e.,

$$hu, hv = 0, \quad \text{and} \quad h\partial_x w,\ h\partial_y w = 0.$$

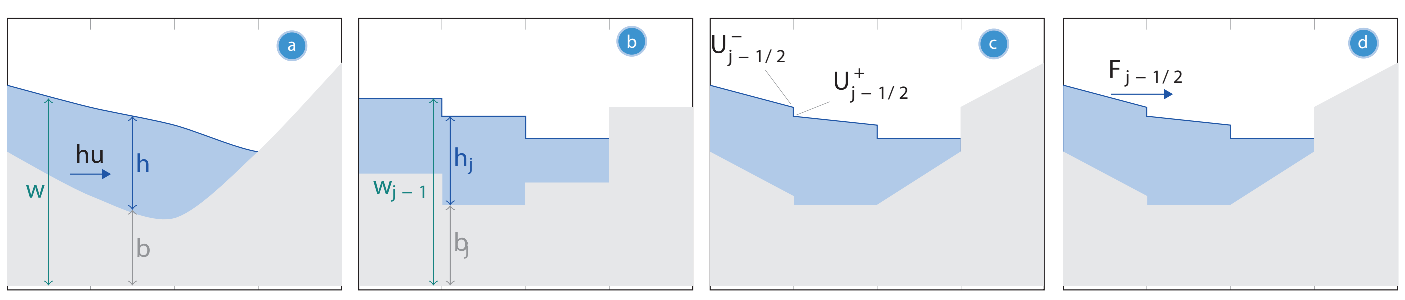
If a numerical scheme is capable of balancing source and numerical flux terms for these two stationary solutions it is called well-balanced and thus preserves the lake at rest and also the still-water equilibrium. Popular methods to solve these equations include the Finite Volume Method (FVM) and the Finite Element Method (FEM).

Second Order for the New CN15 Scheme

There are a lot of numerical schemes devoted to the solution of the SWE. Among them, the Hydrostatic Reconstruction (HR) method is one of the few Finite Volume (FV) schemes that actually

- balances steady states,
- preserves the positivity of the water depth,
- satisfies an entropy inequality,
- and supports a piecewise constant bottom topography.

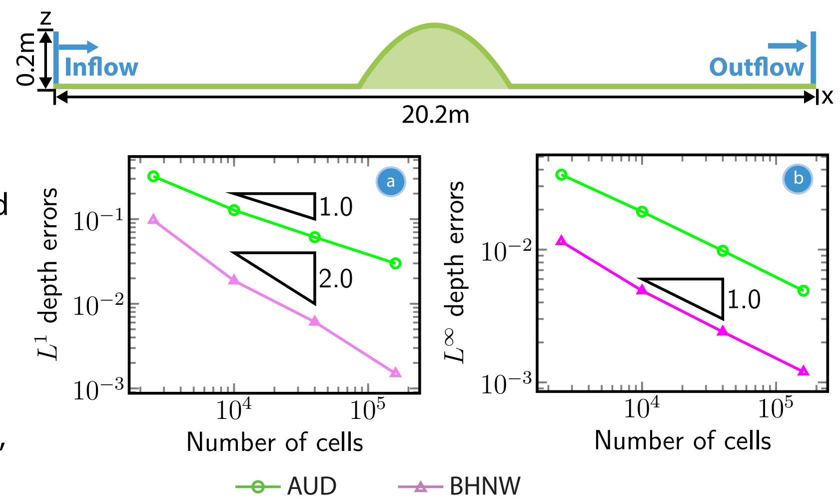
Lately, Chen and Noelle, improved this method to capture downhill and uphill flow more accurately. This scheme, denoted as CN, is first-order, which means that higher-order terms for smooth solutions are neglected. The extension to second-order results in higher efficiency and a better prediction of wave arrival times. Second-order schemes usually suffer from unphysical oscillations, which lead to numerical instabilities in long-term simulations. This insufficiency is overcome by using a minmod slope limiter and a second-order Runge Kutta (RK2) time integration. The second-order scheme is well-balanced and positivity-preserving, if the Courant-Friedrichs-Lewy (CFL) condition holds. In the fully wet cells, the second-order scheme coincides with the second-order scheme of Audusse et al. (2005).
In the figure below, the steps for an FV scheme are shown. First, the physical states (a) in the SWE are discretized by averaging over cells (b). Then, the point values at the interface are reconstructed (c) and used in the flux computation (d). The essential step for second order accuracy is the reconstruction of the point values at the interfaces using a corrected gradient, as illustrated in (c).



Results

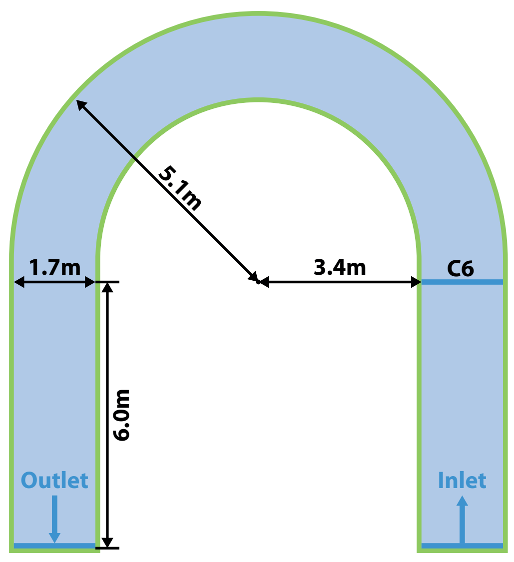
Parabolic Bump

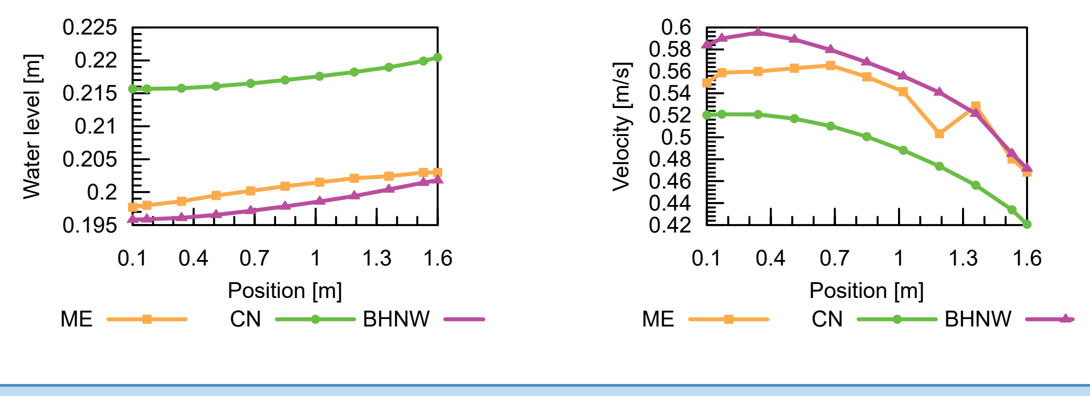
We run simulations to verify the order of the scheme. The basic setup is shown in the right figure. The boundary conditions are set such that they guarantee smooth fluvial flow without any hydraulic jumps. We compute the error of the numerical and the analytical solution for different grid resolutions, or equivalently, different total numbers of cells. The error analysis shows that the new BHNW scheme achieves second-order accuracy for a smooth flow regime in the norm of integrable functions, while the first-order CN scheme shows slower convergence (a). For the maximum norm we can also see an improvement in accuracy compared to the first-order scheme (b). In this fully wet case, the CN scheme and the HR scheme of Audusse et al. (AUD) coincide.



U-Shaped Flume

In order to test the model's capability to simulate the flow in meandering channels, a verification was conducted first for a U-shaped 180° channel. A discharge of 0.18 m^3/s is prescribed at the inflow boundary and a constant water elevation of 0.1875 m at the outflow. Wall boundary conditions are applied along the rasterized flume walls.
In curved channels, the water surface elevation at the outer bank side is higher than the one along the inner bank due to the centrifugal force. The new second-order scheme is named BHNW in the plots below.
A comparison of computed and measured (ME) free surface elevations and depth-averaged velocities for the selected cross section C6 are shown in the figures below.
There are some sources of discrepancies between the measured and the simulated data.
First, in curved channels, there are distinctive characteristics that are completely three dimensional such as secondary currents, which cannot be covered by a depth-averaged model. Second, we use a regular grid, where the cells are not aligned to follow the curvature of the flume and its walls. In the future, we will implement wall boundaries that account for non-aligned walls.





Summary and future work

We have implemented and tested a new scheme for the SWEs. The scheme is second-order accurate in space and time and does not exhibit any oscillations. The second-order accuracy allows us to reduce the discretization error and perform more accurate simulation runs. It is more economical with respect to computation time than a first-order scheme at a fine uniform grid with comparable discretization error. We plan to implement a parallel version of this scheme on graphics processing units (GPUs) to use it also on large real-world cases.

Emmanuel Audusse, François Bouchut, Marie-Odile Bristeau, Rupert Klein, and Benoît Perthame. "A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows". SIAM Journal on Scientific Computing, 25(6):20502065, 2004.

Guoxian Chen and Sebastian Noelle. "A new hydrostatic reconstruction scheme based on subcell reconstructions". SIAM Journal on Numerical Analysis, 55(2):758-784, 2017

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